Rotational optomechanical coupling of a spinning dielectric sphere

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We formulate a non-relativistic Hamiltonian in order to describe how the rotational degrees of freedom of a dielectric sphere and quantized light fields are coupled. Such an interaction is shown to take a form of angular momentum coupling governed by the field angular momentum inside the dielectric. As a specific example, we show that the coupling due to a single whispering gallery mode can lead to precession dynamics and frequency shifts of light.

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Cavity quantum optomechanics has been an active research area investigating quantum phenomena and applications through the interaction between mechanical and optical degrees of freedom [1–5]. In particular, since the mechanical systems such as a dielectric membrane or sphere have masses much greater than that of an atom, the study of quantum optomechanics may test the foundation of quantum theory in macroscopic systems [6]. Typically, the systems considered in cavity quantum optomechanics are deformable cavities. These cavities are subject to radiation pressure pushing their cavity 'walls' apart, which in turn changes the field dynamics. For such systems, the deformation of the cavity is fundamental to the strong coupling between optics and mechanics.

In this paper, we discuss another type of optomechanical coupling that results from the *rotation* of the optical cavity, in which case the coupling remains even in the absence of cavity deformation. Physically, an optical field can affect the rotational motion via the electromagnetic torque exerting to a dielectric object [7–9], and this has been studied in a sequence of experiments [10–14]. The mechanical rotation in turn affects light inside the dielectric, not only because the dielectric changes its orientation, but also due to the *motional-induced* polarization and magnetization [15]. Such a rotational optomechanical coupling could lead to a non-trivial coupled dynamics, and it is then a natural question on how the problem can be formulated self-consistently. In particular, a Hamiltonian formalism of the system would allow a generalization to a fully quantized theory, in which both the optical and mechanical degrees of freedom are quantized.

We remark that there are recent studies beginning to explore quantum effects in optically-trapped dielectric sphere [16–18]. Since the orientation of the levitated dielectric particle is not fixed, rotational dynamics could be possible. However, it remains unclear about the strength of rotational coupling and what fundamental effects can be produced when light is acting on a rotating sphere and vice versa.

The goal of this paper is to formulate a Hamiltonian that can address the rotational dynamics of the coupled dielectric-field system. Specifically, we consider a rigid dielectric sphere with radius R and moment of inertia I, placed in free space. The dielectric constant of the sphere is given by

$$\epsilon(\mathbf{r}) = \begin{cases} n^2, & |\mathbf{r}-\mathbf{r}_0| \le R \\ 1, & \text{otherwise.} \end{cases} \tag{1}$$

We have used the convention $\epsilon_0 = \mu_0 = 1$ (i.e., $c = 1$), and assumed non-magnetic dielectric $\mu = \mu_0$. We have also assumed a non-dispersive and non-absorptive dielectric.

The sphere is free to rotate about any axis, but its center-of-mass (CM) is *fixed* at $\mathbf{r}_0$. In practice the sphere may be confined by an external potential so that the CM of the sphere moves about an equilibrium position. We assume such motion to be negligibly slow and of negligible amplitude, then since for a spherical object, the CM motion does not directly couple with the rotational degrees of freedom [18], our approach here would be a good approximation. The system is specified by the Lagrangian

$$L = \frac{1}{2}I\boldsymbol{\omega}^2 + \int d^3\mathbf{r}\mathcal{L}(\mathbf{r}) \tag{2}$$

where $\boldsymbol{\omega} = (\dot\gamma\sin\beta\cos\alpha - \dot\beta\sin\alpha)\hat{\mathbf{x}} + (\dot\gamma\sin\beta\sin\alpha + \dot\beta\cos\alpha)\hat{\mathbf{y}} + (\dot\alpha + \dot\gamma\cos\beta)\hat{\mathbf{z}}$ is the angular velocity of the sphere, α, β, γ are the 3 Euler angles specifying the orientation of the sphere (we follow the convention in Ref. [19]). $\mathcal{L}$ is the Lagrangian density of the field after eliminating the electronic degrees of freedom of the dielectric. To find $\mathcal{L}$, we go to an *inertial* frame $S'(\mathbf{r})$ in which the dielectric element at $\mathbf{r}$ is instantaneously at rest. Assuming the acceleration of the dielectric does not change its macroscopic properties, the field Lagrangian density at $\mathbf{r}$ in $S'(\mathbf{r})$ is given by the familiar form: $\mathcal{L}' = \frac{1}{2}\left(\epsilon\mathbf{E}'^2 - \mathbf{B}'^2\right)$, where $\mathbf{E}'$ and $\mathbf{B}'$ are the electric and magnetic fields in $S'(\mathbf{r})$, respectively. As the Lagrangian density is Lorentz-invariant, $\mathcal{L}$ can be readily obtained from the Lorentz transformation of the fields from $S'(\mathbf{r})$ to the laboratory frame S. We confine ourselves to a non-relativistic motion of the sphere, so that the velocity $\mathbf{v}(\mathbf{r}) = \boldsymbol{\omega}\times(\mathbf{r}-\mathbf{r}_0)$ of the dielectric element at any point $\mathbf{r}$ satisfies $|\mathbf{v}(\mathbf{r})| \ll c$. Linearizing $\mathcal{L}$ up to first order on $\boldsymbol{\omega}$, the Lagrangian reads

$$L = \frac{1}{2}I\boldsymbol{\omega}^2 + \frac{1}{2}\int d^3\mathbf{r}\left(\epsilon\mathbf{E}^2 - \mathbf{B}^2\right) - \boldsymbol{\omega}\cdot\boldsymbol{\Gamma}, \tag{3}$$

where $\mathbf{\Gamma} = \int d^3\mathbf{r}(\epsilon - 1)(\mathbf{r} - \mathbf{r}_0) \times (\mathbf{E} \times \mathbf{B})$, which takes a form similar to the field angular momentum stored in the sphere. The Lagrangian (3) is a generalization to that of Barton *et al.* [20] and Salamone [21] (in the case $\mu = 1$), which consider a one-dimensional configuration and focus on CM motion of a dielectric slab. Here we will take $\boldsymbol{\omega}$ as a degree of freedom which interacts with the field through the $-\boldsymbol{\omega} \cdot \mathbf{\Gamma}$ term.

The electromagnetic field is specified by the scalar potential $V(\mathbf{r}, t)$ and vector potential $\mathbf{A}(\mathbf{r}, t)$ under the *generalized radiation gauge* $\nabla \cdot [\epsilon(\mathbf{r})\mathbf{A}] = 0$ in the presence of dielectric [22–24], with $\mathbf{E} = -(\partial_t \mathbf{A}) - \nabla V$ and $\mathbf{B} = \nabla \times \mathbf{A}$. For the completeness of our theory, let us first discuss the Euler-Lagrange equations of the system before going to the Hamiltonian.

The first Euler-Lagrange equation for the field is a restatement of the Maxwell equation $\nabla \cdot \mathbf{D} = 0$, which reads: $\nabla \cdot (\epsilon \nabla V) = \nabla \cdot [(\epsilon - 1)\, \mathbf{v} \times (\nabla \times \mathbf{A})]$. This is understood from the fact that the polarization of a moving dielectric element is $\mathbf{P} = (\epsilon - 1)(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ to first order of $v \equiv |\mathbf{v}|$. Under the generalized radiation gauge, V is not a degree of freedom and it is determined by the *instantaneous* values of $\mathbf{A}(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$. We see that V is linear in $\boldsymbol{\omega}$, and vanishes when the dielectric is at rest. The second Euler-Lagrange equation of the field is the wave equation: $\partial_t (\epsilon \partial_t \mathbf{A}) + \nabla \times (\nabla \times \mathbf{A}) = \mathbf{j}$, where $\mathbf{j}$ is a motion-induced (i.e. $\mathcal{O}(v)$) source current density,

$$\mathbf{j} = \partial_t \left[-\epsilon \nabla V + (\epsilon - 1)\, \mathbf{v} \times \mathbf{B}\right] + \nabla \times \mathbf{M}. \tag{4}$$

The wave equation is consistent with the Maxwell equation $\nabla \times \mathbf{B} = \partial_t \mathbf{D} + \nabla \times \mathbf{M}$ in which $\mathbf{M} = -\mathbf{v} \times \mathbf{P}$ is the magnetization of a moving dielectric with the polarization $\mathbf{P}$. From these two Euler Lagrange equations, we see that the motion-induced source terms are fundamental to the sphere-field coupling, without which the rotation of the dielectric sphere cannot affect the time evolution of the field.

The mechanical equation of motion follows from the Euler-Lagrange equations of the Euler angles $(\partial L/\partial \zeta) = (d/dt)(\partial L/\partial \dot{\zeta})$, $\zeta = \alpha, \beta, \gamma$. In terms of $\boldsymbol{\omega}$, we have

$$I \frac{d\boldsymbol{\omega}}{dt} = -\boldsymbol{\omega} \times \mathbf{\Gamma} + \frac{d\mathbf{\Gamma}}{dt}. \tag{5}$$

The two terms on the RHS of Eq. (5) characterize two different types of dynamics of $\boldsymbol{\omega}$. The first term describes a precession about the $\mathbf{\Gamma}$ axis with a frequency $|\mathbf{\Gamma}|/I$, which keeps the magnitude of $\boldsymbol{\omega}$ unchanged [25]. On the other hand, the second term may change the magnitude of $\boldsymbol{\omega}$ along the $\mathbf{\Gamma}$ axis. We remark that Eq. (5) is consistent with the conservation of total angular momentum $\frac{d}{dt}(I\boldsymbol{\omega} + \mathbf{J}_F) = 0$ which follows from the rotational invariance of the Lagrangian (3). To zeroth order in $\boldsymbol{\omega}$, the total field angular momentum reads $\mathbf{J}_F = \int d^3\mathbf{r}(\mathbf{r} - \mathbf{r}_0) \times (\mathbf{E} \times \mathbf{B})$ [26].

We now turn to the Hamiltonian defined from L by

$$H \equiv \sum_{\zeta = \alpha, \beta, \gamma} \dot{\zeta} p_\zeta + \int \mathbf{\Pi} \cdot (\partial_t \mathbf{A})\, d^3\mathbf{r} - L, \tag{6}$$

where $\mathbf{\Pi}(\mathbf{r}, t) \equiv [\partial \mathcal{L}/\partial(\partial_t \mathbf{A})]$ is the field canonical momentum density, and $p_\zeta \equiv (\partial L/\partial \dot{\zeta})$ are the canonical momenta conjugate to the Euler angles. We introduce a *canonical angular momentum* $\mathbf{J}$ in terms of p_ζ [27]:

$$\begin{aligned} J_x &= -\cot\beta \cos\alpha p_\alpha - \sin\alpha p_\beta + \csc\beta \cos\alpha p_\gamma, \\ J_y &= -\cot\beta \sin\alpha p_\alpha + \cos\alpha p_\beta + \csc\beta \sin\alpha p_\gamma, \\ J_z &= p_\alpha. \end{aligned} \tag{7}$$

Explicitly, $\mathbf{\Pi}(\mathbf{r}, t)$ and $\mathbf{J}$ are given by

$$\mathbf{\Pi} = -\epsilon \mathbf{E} - (\epsilon - 1)(\mathbf{v} \times \mathbf{B}) = -\mathbf{D}, \tag{8}$$

$$\mathbf{J} = I\boldsymbol{\omega} - \mathbf{\Gamma}. \tag{9}$$

Note that $\mathbf{\Pi}$ is transverse as $\nabla \cdot \mathbf{D} = 0$, and $\mathbf{J}$ differs from the kinetic angular momentum $I\boldsymbol{\omega}$ for non-zero fields. The explicit expression of the Hamiltonian (6) reads

$$H = \frac{(\mathbf{J} + \mathbf{\Gamma}')^2}{2I} + \frac{1}{2} \int d^3\mathbf{r} \left(\frac{\mathbf{\Pi}^2}{\epsilon} + \mathbf{B}^2 \right), \tag{10}$$

with $\mathbf{\Gamma}'$ given by

$$\mathbf{\Gamma}' = -\int d^3\mathbf{r} \left(\frac{\epsilon - 1}{\epsilon} \right) \left[(\mathbf{r} - \mathbf{r}_0) \times (\mathbf{\Pi} \times \mathbf{B}) \right]. \tag{11}$$

This Hamiltonian takes a form similar to the minimal-coupling Hamiltonian in electrodynamics, with $\mathbf{\Gamma}'$ somehow playing the role of vector potential in the kinetic energy term. We note that in writing Eq. (10), we have neglected field-dependent terms that are quadratic in $\mathbf{v}$. These terms resemble the kinetic energy of the sphere, and contribute to a correction of the moment of inertia I due to the field. Such a correction is typically very small compared with I for fields well below the dielectric breakdown of the sphere.

With the classical Hamiltonian (10), the canonical quantization of the system is readily achieved by promoting the dynamical variables ζ, p_ζ $(\zeta = \alpha, \beta, \gamma)$, $\mathbf{A}(\mathbf{r})$ and $\mathbf{\Pi}(\mathbf{r})$ into operators by postulating the commutation relations:

$$[\zeta, p_\eta] = i\hbar \delta_{\zeta\eta} \tag{12}$$

$$[A_i(\mathbf{r}), \Pi_j(\mathbf{r}')] = i\hbar \delta^\epsilon_{ij}(\mathbf{r}, \mathbf{r}') \tag{13}$$

where $\delta^\epsilon_{ij}(\mathbf{r}, \mathbf{r}')$ is a *generalized transverse δ-function* in the presence of dielectric [22]. From Eq. (7), the commutation relations of the Euler angles α, β, γ imply that $\mathbf{J}$ forms a quantum rigid rotor [19], which includes the angular momentum commutation relation $[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$. The quantum Hamiltonian takes the same expression as Eq. (10), but with $\mathbf{\Gamma}'$ defined in Eq. (11) symmetrized, i.e. with the bracketed term in the integrand replaced by $[(\mathbf{r} - \mathbf{r}_0) \times (\mathbf{\Pi} \times \mathbf{B} - \mathbf{B} \times \mathbf{\Pi})]/2$.

In order to discuss field excitations in Fock space, we project the field operators onto a complete set of mode functions, namely the TE and TM mode functions (using spherical coordinates with origin at $\mathbf{r}_0$) $\mathbf{\Psi}_{lm}(k, \mathbf{r}) = u_l^{(E)}(k, r)\mathbf{X}_{lm}(\theta, \phi)$ and

$\mathbf{\Phi}_{lm}(k,\mathbf{r}) = (i/k)\nabla\times[u_l^{(M)}(k,r)\mathbf{X}_{lm}(\theta,\phi)]$ respectively, with $\mathbf{X}_{lm}(\theta,\phi)$ being the vector spherical harmonics [28], and $u_l^{(E)}(k,r)$ and $u_l^{(M)}(k,r)$ are radial functions subject to appropriate boundary conditions across $r=R$: $\nabla\cdot[\epsilon(r)\mathbf{\Psi}_{lm}(k,\mathbf{r})] = \nabla\cdot[\epsilon(r)\mathbf{\Phi}_{lm}(k,\mathbf{r})] = 0$ [29]. Substituting the normal-mode expansion into the Hamiltonian (we take $\hbar = 1$ from here on),

$$H = \frac{(\mathbf{J}+\mathbf{\Gamma}')^2}{2I} + H_F \tag{14}$$

where $H_F = \int dk \sum_{l,m} \omega_k [a_{lm}^\dagger(k)a_{lm}(k) + b_{lm}^\dagger(k)b_{lm}(k)]$ is the field Hamiltonian with constant terms removed, $a_{lm}(k)$ and $b_{lm}(k)$ are the annihilation operators for TE and TM mode photons with quantum numbers (k,l,m) and a frequency $\omega_k = ck$, respectively.

Equations (10) and (14) are main results of this paper. It is important to note that the form of $\mathbf{\Gamma}'$ [Eq. (11)] is very similar to the field angular momentum stored in the dielectric, apart from some proportionality constant. Therefore, approximately speaking, the first term of the Hamiltonian (10) and (14) represents an angular momentum coupling, i.e., the interaction corresponds to an exchange of angular momenta between the field and the sphere. In particular, if the field is localized inside the sphere, we expect that $\mathbf{\Gamma}'$ should become a good approximation to the *total* field angular momentum, up to a multiplicative constant.

As an illustrative example of the rotational optomechanical coupling, we apply the Hamiltonian (14) to a configuration in which photons occupy a whispering-gallery mode (WGM). In this case photons can be confined inside the dielectric cavity with a long life time due to multiple total internal reflections. For simplicity, we consider that the field excitation is dominantly contributed by TE mode photons with frequencies $\omega_k \approx \omega_0$, where $\omega_0 = ck_0$ is a resonant frequency of a TE WGM. Assuming that such a WGM has a narrow line width κ_c and ω_0 is well separated from all other TM mode frequencies, it is sufficient to include TE modes only in the Hamiltonian. Furthermore, since the optical quality factor of the spherical cavity is typically very high, i.e. $Q = k_0/\kappa_c \gg 1$, it is instructive to consider the dynamics of the system within a time scale short compared with κ_c^{-1}. In this regime, the leakage of WGM photons is negligible. Then the field Hamiltonian associated with TE WGMs with orbital quantum number l is given by $H_F \approx \sum_{m=-l}^{l} \omega_0 c_m^\dagger c_m$, where c_m is the cavity mode operator [30]

$$c_m = \sqrt{\frac{\kappa_c}{\pi}} \int dk \frac{a_{lm}(k)}{k - k_0 + i\kappa_c}. \tag{15}$$

Here the index l for c_m is suppressed for compactness.

With the help of cavity mode c_m operators, the $\mathbf{\Gamma}'$ operator contributed by the TE WGM is approximately given by,

$$\mathbf{\Gamma}' \approx \Lambda \sum_{mm'} \left(\int d\Omega Y_{lm'}^* \mathbf{L} Y_{lm} \right) c_{m'}^\dagger c_m \equiv \Lambda \mathbf{S}, \tag{16}$$

where $\mathbf{L} = -i(\mathbf{r}-\mathbf{r}_0)\times\nabla$, $Y_{lm}(\theta,\phi)$ are spherical harmonics and

$$\Lambda = \pi\kappa_c \int_0^R dr (\epsilon - 1) r^2 |u_l^{(E)}(k_0,r)|^2 \tag{17}$$

is a dimensionless parameter determined by the mode amplitude inside the sphere. Numerical calculations with the parameters $R = 10$ μm, $n^2 = 2.31$, $l = 120$, $k_0 = 2\pi/(743.25\text{ nm})$ leads to $\Lambda = 1.12$.

In writing Eq. (16) we have employed the rotating-wave approximation (RWA), so that fast oscillating terms such as $c_m^\dagger c_{m'}^\dagger$ are dropped. However, these terms are responsible for photon generation in dynamical Casimir effect [31], and they should be retained if such a quantum effect become significant, for example, when $\boldsymbol{\omega}(t)$ is rapidly changing with time. We also remark that the angular integral in Eq. (16) gives the selection rules for the rotational coupling. Together with $[c_m, c_{m'}^\dagger] = \delta_{mm'}$, we see that $\mathbf{S}$ defined in Eq. (16) satisfies the angular momentum commutation relations $[S_i, S_j] = i\epsilon_{ijk}S_k$.

By combining Eq. (14), the interaction with the TE WGM leads to a Hamiltonian (in a rotating frame where H_F is eliminated),

$$H_r = \Lambda\frac{(\mathbf{J}+\mathbf{S})^2}{2I} + (1-\Lambda)\frac{\mathbf{J}^2}{2I} + \Lambda(\Lambda-1)\frac{\mathbf{S}^2}{2I}. \tag{18}$$

which describes a coupling between the angular momenta $\mathbf{J}$ and $\mathbf{S}$, with coupling strengths characterized by Λ. In addition, the Heisenberg equations of motion for $\mathbf{S}$ and $\boldsymbol{\omega}$ (noting that $\mathbf{J} = I\boldsymbol{\omega} - \Lambda\mathbf{S}$) become

$$\dot{\mathbf{S}} = \Lambda\boldsymbol{\omega}\times\mathbf{S} \quad \text{and} \quad I\dot{\boldsymbol{\omega}} = \Lambda(\Lambda-1)(\boldsymbol{\omega}\times\mathbf{S}) \tag{19}$$

respectively, indicating that the optical ($\mathbf{S}$) and mechanical angular momentum ($I\boldsymbol{\omega}$) precess about each other in the rotating frame. Moreover, for the coherent time scale concerned here (i.e., with negligible cavity field decay), the gallery mode photons cannot change the magnitude of $\boldsymbol{\omega}$, and the sphere can only precess about the instantaneous $\mathbf{S}$-axis.

An order-of-magnitude estimate of the mechanical precession rate can be made by $\Lambda(\Lambda-1)\langle\mathbf{S}\rangle/I \approx (n^2-1)N\hbar l/\rho R^5$, where N is the number of cavity photons, and ρ is the mass density of the sphere. Under the same numerical parameters above, and assuming $N \approx 10^5$ before dielectric breakdown, the precession rate would be on the order of 10^{-5} Hz. If the sphere spins *coherently* at a *macroscopic* rate and the WGM field inside the cavity is sufficiently weak such that $I|\langle\boldsymbol{\omega}\rangle| \gg |1-\Lambda||\langle\mathbf{S}\rangle|$, then $\boldsymbol{\omega} \approx \langle\boldsymbol{\omega}\rangle$ behaves approximately as a constant (classical) vector. In this case the field dynamics described by $\mathbf{S}$ is governed by an effective Hamiltonian $H_{\text{eff}} = \Lambda\langle\boldsymbol{\omega}\rangle\cdot\mathbf{S}$, which has the same form as that of a magnetic moment in an external magnetic field. Therefore photons initially occupying the cavity mode m experiences a *Zeeman-type* frequency shift of $m\Lambda\langle\omega_z\rangle$. We note that the frequency shift should be resolved from the cavity linewidth (i.e.

$\Lambda\langle\omega_z\rangle \gtrsim \kappa_c$) in order to be observable. With an optical field with quality factor of $Q \sim 10^{10}$, such a condition would require the dielectric sphere to spin at a rate of 1 kHz.

To conclude, we have established a non-relativistic Lagrangian and a Hamiltonian for a three-dimensional sphere-field system in the $v \ll c$ regime. By including the motion-induced polarization and magnetization possessed by the dielectric sphere, we have self-consistently determined how a spinning dielectric sphere and quantized light fields are coupled. The sphere-field interaction is described by the Hamiltonian (10) and (14) as a coupling of the canonical angular momentum $\mathbf{J}$ to the quantity $\mathbf{\Gamma}'$, which is proportional to the field angular momentum stored in the dielectric sphere. We further illustrated the rotational coupling using a WGM description of the Hamiltonian, and identified the constant Λ that determines the sphere-field coupling strength. Within the coherent time scale of the WGM photons, we have shown that the optical and mechanical angular momentum precesses about each other, and the degenerate WGM multiplets would experience a Zeeman-type splitting under a strong mechanical rotation. Although such effects are weak in general, they reveal fundamental features arising from the rotational degrees of freedom of the fields and the sphere. Our work here should provide a framework to further explore quantum phenomena and applications of such rotational optomechanical coupling.

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$$u_l^{(\nu)}(k,r) = \begin{cases} \frac{k}{\sqrt{\pi}} T_l^{(\nu)}(k) j_l(nkr), & r \le R \\ \frac{k}{\sqrt{\pi}} \left[R_l^{(\nu)}(k) h_l^{(1)}(kr) + h_l^{(2)}(kr) \right], & r > R, \end{cases}$$

where j_l, $h_l^{(1)}$ and $h_l^{(2)}$ are the usual spherical bessel and hankel functions (of first and second kind), respectively. See, for example, J. Ji, C. Lee, J. Noh and W. Jhe, J. Phys. B: At. Mol. Opt. Phys. **33** 4821, (2000).
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